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Singular Perturbation of Domains and the Semilinear Elliptic Equation, II

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1. INTRODUCTION

In this paper, we will deal with a singularly perturbed domain $\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta)$ (Fig. 1) with a small parameter $\zeta > 0$ and will investigate the behavior of the solutions and their structure of the following semilinear elliptic boundary value problem (1.1) for $\Omega = \Omega(\zeta)$ when $\zeta > 0$ is small:

$$\begin{aligned} \Delta v + f(v) &= 0 \qquad \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \nu} &= 0 \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Delta = \sum_{j=1}^{n} (\partial^2/\partial x_j^2)$ is the Laplace operator and the nonlinear term f is a real valued smooth function on $\mathbb{R}$.

In our previous work [8], we have characterized the behavior of a certain class of the solutions $\{v_\zeta\}_{\zeta > 0}$ of (1.1) on the domain $\Omega(\zeta)$ in Fig. 1. More precisely, if v_ζ is an arbitrary solution of (1.1) for $\Omega = \Omega(\zeta)$ which satisfies

$$\begin{aligned} &\lim_{\zeta \to 0} \|v_\zeta - a_i\|_{L^2(D_i)} = 0, \\ &f(a_i) = 0, f'(a_i) < 0 \qquad (i = 1, 2), \end{aligned} \tag{1.2}$$

then the behavior of v_ζ for small $\zeta > 0$ in the singular portion $Q(\zeta)$ is approximated by that of some solution V of the following two point boundary value problem of the ordinary differential equation

$$\frac{d^2 V}{dz^2} + f(V) = 0 \qquad \text{for} \quad z \in L, \tag{1.3}$$

with the boundary condition $V(z) = a_i$ for $z \in \partial L \cap \partial D_i$ $(i = 1, 2)$ where L is a one-dimensional line segment $L \equiv \bigcap_{\zeta > 0} \overline{Q(\zeta)}$ and z is the variable along

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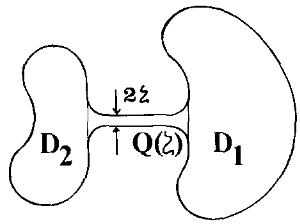

FIGURE 1

L (see Fig. 2). We also related the stability of v_ζ in (1.1) to that of V in (1.3), i.e., we proved that if the linearized first eigenvalue of V in (1.3) is positive (resp. negative), then that of v_ζ in (1.1) is positive (resp. negative) for small $\zeta > 0$. Conversely, by choosing a special nonlinear term f so that the boundary value problem (1.3) has both unstable solution V_1 and stable one V_2, we have constructed unstable solution $v_\zeta^{(1)}$ and stable one $v_\zeta^{(2)}$ of (1.1) for small $\zeta > 0$, where each $v_\zeta^{(i)}$ is approximated by V_i on $Q(\zeta)$ $(i = 1, 2)$ while $v_\zeta^{(1)}$ and $v_\zeta^{(2)}$ have almost the same behavior in $D_1 \cup D_2$. In view of these results, the moving portion $Q(\zeta)$ which becomes thinner as $\zeta \to 0$ does not lose influence over (1.1) (i.e., the structure of the solutions) for $\Omega = \Omega(\zeta)$ while it loses the volume.

In this paper, we deal with a general solution v_ζ of (1.1) (for $\Omega = \Omega(\zeta)$, $\zeta > 0$ small) on which we do not impose a condition such as (1.2) concerning the behavior in $D_1 \cup D_2$ and characterize the behavior of this solution. We will determine the behavior of the linearized first eigenvalue of v_ζ which we only gave a rough characterization in [8]. More precisely speaking, under the condition $n \geqslant 3$, we will prove that for small $\zeta > 0$, any solution v_ζ of (1.1) (for $\Omega = \Omega(\zeta)$) is approximated by some w_i $(\in C^\infty(\bar{D}_i))$ in D_i $(i = 1, 2)$ and is approximated by some V $(\in C^\infty(L))$ in $Q(\zeta)$ where w_i is a solution of (1.1) for $\Omega = D_i$ and V is a solution of the ordinary differential equation (1.3) with the boundary (compatibility) condition

$$V|_{\partial L \cap \partial D_i} = w_i|_{\partial L \cap \partial D_i} \qquad (i = 1, 2) \tag{1.4}$$

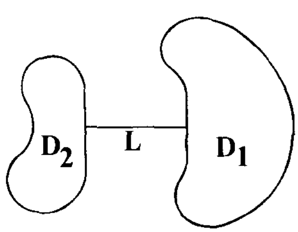

FIGURE 2

and we also prove that the linearized first eigenvalue of v_ζ in $\Omega(\zeta)$ is characterized as

$$\mu_1(v_\zeta, \Omega(\zeta)) \sim \min\{\mu_1(w_1, D_1), \mu_1(w_2, D_2), \lambda_1(V)\}, \tag{1.5}$$

where $\mu_1(w, \Omega)$ is the linearized first eigenvalue of w in (1.1) in Ω and $\lambda_1(V)$ is the linearized first eigenvalue of V in (1.3) with the Dirichlet boundary condition on ∂L.

In [8], it was the key point to prove the uniform convergence of the solution in the singular portion which is a moving neighborhood of the point p_i of the domain $\Omega(\zeta)$. To do so, we used a comparison function which estimates $|v_\zeta(x) - a_i|$ in this portion and was a linear combination of the constant function and the Green function $1/|x-p_i|^{n-2}$ $(n \geqslant 3)$. But, to apply the comparison theorem, we assumed $f'(a_i) < 0$ $(i = 1, 2)$ there. Therefore we could not deal with general solutions without any conditions such as (1.2). In this paper, to remove this type of difficulty, we use two radially symmetric solutions $A_1(M^{1/2}|x-p_i|)$, $A_2(M^{1/2}|x-p_i|)/|x-p_i|^{n-2}$ (A_1, A_2: Bessel function or Neumann function) of the following Helmholtz equation in place of harmonic functions to construct comparison functions,

$$\Delta\phi + M\phi = 0 \qquad (M > 0, \text{ large}). \tag{1.6}$$

By this device, we can also obtain a rather exact asymptotic behavior of the linearized eigenvalue (1.5) by characterizing the first eigenfunction in $\Omega(\zeta)$ globally. In view of these results, we can get insight into the structure of the solutions. These are the main parts of this paper (Sections 3–8). In this paper, before the main result, we deal with a rather rough situation in Section 2, i.e., we consider a perturbed domain $\Omega(\zeta)$ $(\zeta > 0)$ of a set $D_1 \cup D_2 \cup \cdots \cup D_N$ (N connected components) in a very weak sense, as follows: $\lim_{\zeta \to 0} \mathrm{Vol}(\Omega(\zeta) - \bigcup_{i=1}^N D_i) = 0$. We show that if there exists a solution w_i of (1.1) for $\Omega = D_i$ such that $\mu_1(w_i, D_i) > 0$ for each i $(1 \leqslant i \leqslant N)$, there exists a stable solution of (1.1) for $\Omega = \Omega(\zeta)$ which behaves like w_i in D_i for each i. See below for the definition of $\mu_k(w_i, D_i)$. This is a generalization of [8, Theorem 1]. This existence theorem assures that the characterization theorem (the main result) in Section 3 contains a lot of examples. In the proof of Theorem 1, we essentially use Matano's result in [9]. Therefore we review it in Section 8 in the form which is suitable to our situation.

We give the definition of the stability of the solution of (1.1) and notations of the linearized eigenvalues.

DEFINITION 1. A solution v of (1.1) is said to be stable if given any constant $\varepsilon > 0$, there exists a constant $\delta > 0$, such that $\|u(t, \cdot) - v(\cdot)\|_{L^\infty(\Omega)} \leqslant \varepsilon$ $(0 < t < \infty)$ for any $B \in C^0(\bar{\Omega})$ satisfying $\|B - v\|_{L^\infty(\Omega)} \leqslant \delta$, where $u(t, x)$ is a solution of the following semilinear diffusion equation (1.6) with the initial condition $u(0, \cdot) = B$:

$$\begin{aligned} \partial u/\partial t &= \Delta u + f(u) \qquad (t, x) \in (0, \infty) \times \Omega, \\ \partial u/\partial \nu &= 0 \qquad\qquad\quad (t, x) \in (0, \infty) \times \partial\Omega. \end{aligned} \tag{1.7}$$

We say that v is unstable if v is not stable

Notation 1. We denote by $\{\mu_k(v, \Omega)\}_{k=1}^{\infty}$ and $\{\varphi_k(v, \Omega)\}_{k=1}^{\infty}$, respectively, the sequence of the eigenvalues (arranged in increasing order) and the complete system of the corresponding orthonormalized eigenfunctions of the following eigenvalue problem:

$$\begin{aligned} \Delta\varphi + f'(v)\varphi + \mu\varphi &= 0 \qquad \text{in} \quad \Omega, \\ \partial\varphi/\partial\nu &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.8}$$

PROPOSITION 1.1 [6]. *If* $\mu_1(v, \Omega) > 0$ *(resp.* $\mu_1(v, \Omega) < 0$*),* v *is stable (resp. unstable).*

All the functions that we consider in this paper are real valued.

2. EXISTENCE OF STABLE SOLUTIONS

In this section, we prove an existence theorem for stable solutions where $\Omega(\zeta)$ $(\zeta > 0)$ is a (wildly) perturbed domain from the set which has N connected components $(D_1, D_2, ..., D_N)$. This domain perturbation can be very wild because the condition (II.1)(ii) is not very strong and it contains the one in Section 3 as a special case when $N = 2$. We prove that if there exists a stable (in the sense of the linearized eigenvalue) solution w_i (i.e., $\mu_1(w_i, D_i) > 0$), there exists a stable solution v_ζ of the following equation (2.1) which approximates w_i in D_i for small $\zeta > 0$ for each i:

$$\begin{aligned} \Delta v + f(v) &= 0 \qquad \text{in} \quad \Omega(\zeta), \\ \frac{\partial v}{\partial \nu} &= 0 \qquad \text{on} \quad \partial\Omega(\zeta). \end{aligned} \tag{2.1}$$

(II.1) $D_1, D_2, ..., D_N$ are bounded domains in $\mathbb{R}^n$ $(n \geqslant 2)$ where each D_j has a smooth boundary ∂D_j and $\bar{D}_i \cap \bar{D}_j = \varnothing$ $(i < j)$. For each $\zeta > 0$, $\Omega(\zeta)$ is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega(\zeta)$ with the following two properties:

(i) $\Omega(\zeta_1) \supset \Omega(\zeta_2) \supset \bigcup_{i=1}^{N} D_i$ for any $\zeta_1 > \zeta_2 > 0$.

(ii) $\lim_{\zeta \to 0} \mathrm{Vol}(\Omega(\zeta) - \bigcup_{i=1}^{N} D_i) = 0$.

We impose the following condition on the nonlinear term f:

(II.2) $f \in C^{\infty}(\mathbb{R})$, $\overline{\lim}_{\xi \to \infty} f(\xi) < 0$, $\underline{\lim}_{\xi \to -\infty} f(\xi) > 0$.

We assume the following situation.

(II.3) For each i $(1 \leqslant i \leqslant N)$, there exists a function $w_i \in C^{\infty}(\bar{D}_i)$ such that $\Delta w_i + f(w_i) = 0$ in D_i, $\partial w_i/\partial \nu = 0$ on ∂D_i with $\mu_1(w_i, D_i) > 0$.

We have the following theorem.

THEOREM 1. *We assume* (II.1), (II.2), *and* (II.3). *For each* $\zeta > 0$ *there exists at least one stable solution* v_ζ *of* (2.1) *such that the following property* (2.2) *holds*:

$$\begin{aligned} \lim_{\zeta \to 0} \|v_\zeta - w_i\|_{L^2(D_i)} = 0 \qquad (1 \leqslant i \leqslant N), \\ \lim_{\zeta \to 0} v_\zeta = w_i \text{ in } C^{\infty}(\overline{D_i(\eta)}) \text{ for any } \eta > 0 \qquad (1 \leqslant i \leqslant N). \end{aligned} \tag{2.2}$$

where $D_i(\eta) \equiv \{x \in D_i \mid \operatorname{dis}(x, \Omega(\eta) - D_i) > \eta\}$.

Proof of Theorem 1. Put $\underline{M} \equiv \inf\{\xi \in \mathbb{R} \mid f(\xi) = 0\}$ and $\bar{M} \equiv \sup\{\xi \in \mathbb{R} \mid f(\xi) = 0\}$. Arbitrary solution of (2.1) (if it exists) takes its values in the interval $[\underline{M}, \bar{M}]$ by condition (II.2); therefore, we may consider the equation given by replacing f by the nonlinear term $\tilde{f} \in C^{\infty}(\mathbb{R})$ such that $\tilde{f}(\xi) = f(\xi)$ for $\xi \in [\underline{M} - 1, \bar{M} + 1]$), $\tilde{f}(\xi) = f(\bar{M} + 2)$ for $\xi \in [\bar{M} + 2, \infty)$, and $\tilde{f}(\xi) = f(\underline{M} - 2)$ for $\xi \in (-\infty, \underline{M} - 2]$). Therefore without loss of generality, we assume in this proof that $\operatorname{supp}(\partial f(\xi)/\partial \xi) \subset [\underline{M} - 2, \bar{M} + 2]$. (Remark that $f(\xi) < 0$ for $\xi > \bar{M}$ and $f(\xi) > 0$ for $\xi < \underline{M}$.) Hereafter we put $Q(\zeta) \equiv \Omega(\zeta) - \bigcup_{i=1}^{N} D_i$. Let A be a function in $C^{\infty}(\mathbb{R}^n)$ such that $A(x) = w_i(x)$ for $x \in D_i$ $(1 \leqslant i \leqslant N)$, $\operatorname{supp}(\operatorname{grad} A)$ is compact in $\mathbb{R}^n$, and $\underline{M} \leqslant A(x) \leqslant \bar{M}$ in $\mathbb{R}^n$. We define a functional T_ζ on $H^1(\Omega(\zeta))$ and a closed subset $E(\delta, \zeta)$ of $C^1(\overline{\Omega(\zeta)}) \cap C^2(\Omega(\zeta))$ as

$$\begin{aligned} T_\zeta(v) \equiv {} & \sum_{i=1}^{N} \int_{D_i} \left(\frac{1}{2} |\nabla(v - w_i)|^2 - \int_{A(x)}^{v(x)} (f(\xi) - f(A(x)))\, d\xi \right) dx \\ & + \int_{Q(\zeta)} \left(\frac{1}{2} |\nabla v|^2 - \int_{A(x)}^{v(x)} f(\xi)\, d\xi \right) dx, \end{aligned} \tag{2.3}$$

$$\begin{aligned} E(\delta, \zeta) \equiv {} & \{v \in C^1(\overline{\Omega(\zeta)}) \cap C^2(\Omega(\zeta)) \mid T_\zeta(v) \leqslant T_\zeta(A) + \delta^3, \\ & \|v - w_i\|_{L^2(D_i)} \leqslant \delta \ (1 \leqslant i \leqslant N),\ \underline{M} - \delta \leqslant v(x) \leqslant \bar{M} + \delta \ (x \in \Omega(\zeta))\}. \end{aligned}$$

Our method is to find a positive function $\delta(\zeta)$ defined on some interval $(0, \bar{\zeta})$ such that $\lim_{\zeta \to 0} \delta(\zeta) = 0$ and $E(\delta, \zeta)$ is invariant under the following semilinear diffusion equation (2.4) when δ belongs to the interval $[\delta(\zeta), 2\delta(\zeta)]$, that is, for any given B in $E(\delta, \zeta)$ such that $\delta(\zeta) \leqslant \delta \leqslant 2\delta(\zeta)$, there exists a global solution $u_\zeta(t,x)$ of (2.4) such that $u_\zeta(t,\cdot)\in E(\delta,\zeta)$ for $t\in(0,\infty)$:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(u) && (t,x)\in(0,\infty)\times\Omega(\zeta), \\ \frac{\partial u}{\partial \nu} &= 0 && (t,x)\in(0,\infty)\times\partial\Omega(\zeta), \\ u(0,x) &= B(x) && x\in\Omega(\zeta). \end{aligned} \tag{2.4}$$

Then we can apply Matano's Theorem [9, Theorem 4.2] to the set $E(\delta(\zeta),\zeta)$. In fact, by applying Proposition 8.2 for $E=E(\delta(\zeta),\zeta)$, $E_m=E(((m+1)/m)\,\delta(\zeta),\zeta)$ $(m\geqslant 1)$ and $\Omega=\Omega(\zeta)$, we get a stable solution in $E(\delta(\zeta),\zeta)$.

By the aid of the comparison–existence theorem and (II.2), it is easily seen that for any $B\in C^0(\overline{\Omega(\zeta)})$ such that $\underline{M}-\delta\leqslant B(x)\leqslant\bar{M}+\delta$ in $\Omega(\zeta)$, there exists a global solution $u_\zeta(t,x)$ of (2.4) such that $\underline{M}-\delta\leqslant u_\zeta(t,x)\leqslant \bar{M}+\delta$ for $(t,x)\in[0,\infty)\times\Omega(\zeta)$. For this u_ζ, we have the following inequality.

LEMMA 2.1 (Energy inequality).

$$T_\zeta(u_\zeta(t,\cdot))\leqslant T_\zeta(B) \qquad (t\geqslant 0). \tag{2.5}$$

Proof of Lemma 2.1. T_ζ is also written as

$$\begin{aligned} T_\zeta(u_\zeta(t,\cdot)) &= \int_{\Omega(\zeta)}\left(\frac{1}{2}|\nabla u_\zeta|^2-\int_A^{u_\zeta} f(\xi)\,d\xi\right)dx \\ &\quad + \sum_{i=1}^N\int_{D_i}\left(-\nabla u_\zeta\,\nabla w_i+\frac{1}{2}|\nabla w_i|^2+f(A)(u_\zeta-A)\right)dx. \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{\partial}{\partial t}T_\zeta(u_\zeta(t,\cdot)) &= \int_{\Omega(\zeta)}\left(\nabla\left(\frac{\partial u_\zeta}{\partial t}\right)\nabla u_\zeta - f(u_\zeta)\frac{\partial u_\zeta}{\partial t}\right)dx \\ &\quad + \sum_{i=1}^N\int_{D_i}\left(-\nabla\left(\frac{\partial u_\zeta}{\partial t}\right)\nabla w_i+f(A)\frac{\partial u_\zeta}{\partial t}\right)dx \\ &= -\int_{\Omega(\zeta)}\left(\frac{\partial u_\zeta}{\partial t}\right)(\Delta u_\zeta+f(u_\zeta))\,dx \\ &\quad + \sum_{i=1}^N\int_{D_i}(\Delta w_i+f(w_i))\frac{\partial u_\zeta}{\partial t}\,dx \\ &= -\int_{\Omega(\zeta)}\left(\frac{\partial u_\zeta}{\partial t}\right)^2 dx\leqslant 0. \end{aligned}$$

By Lemma 2.1, we have $T_\zeta(u_\zeta(t,\cdot))\leqslant T_\zeta(A)+\delta^3$ $(0\leqslant t<\infty)$. Hereafter we will estimate $\|u_\zeta(t,\cdot)-w_i\|_{L^2(D_i)}$ from above by use of $T_\zeta(u_\zeta(t,\cdot))$. The following inequality is the key lemma.

LEMMA 2.2. *There exists a positive constant* $\delta_1>0$ *such that*

$$\min_{1\leqslant i\leqslant N}\left(\frac{1}{2},\frac{\mu_1(w_i,D_i)}{4}\right)\sum_{i=1}^{N}(\|U\|_{L^2(D_i)})^2$$
$$\leqslant T_\zeta(A+U)+\int_{\underline{M}-2}^{\bar{M}+2}|f(\xi)|\,d\xi\cdot\mathrm{Vol}(Q(\zeta)) \tag{2.6}$$

for any $U\in C^1(\overline{\Omega(\zeta)})$ *such that* $\|U\|_{L^2(D_i)}\leqslant\delta_1$ $(1\leqslant i\leqslant N)$ *and* $\underline{M}-2\leqslant A(x)+U(x)\leqslant\bar{M}+2$ *for* $x\in\Omega(\zeta)$.

Proof of Lemma 2.2. We use the eigenfunction expansion of U in each D_i. We write $\varphi_{i,k}$ in place of $\varphi_k(w_i,D_i)$ for brevity:

$$U_{i,q}(t,x)\equiv\sum_{k=1}^{q}(U,\varphi_{i,k})_{L^2(D_i)}\varphi_{i,k}(x)\qquad(x\in D_i,\ 1\leqslant i\leqslant N).$$

Now we estimate the first term of (2.3) from below by the following decomposition:

$$\int_{D_i}\left(\frac{1}{2}|\nabla U|^2-\int_{A(x)}^{A(x)+U(x)}(f(\xi)-f(A(x))\,d\xi\right)dx$$
$$=\int_{D_i}\left(\frac{1}{2}|\nabla U_{i,q}|^2-\int_0^{U_{i,q}(t,x)}(f(w_i+\xi)-f(w_i))\,d\xi\right)dx$$
$$+\int_{D_i}\left(\frac{1}{2}|\nabla(U_i-U_{i,q})|^2-\int_{U_{i,q}(t,x)}^{U_i(t,x)}(f(w_i+\xi)-f(w_i+U_{i,q}))\,d\xi\right)dx$$
$$+\int_{D_i}(\nabla(U_i-U_{i,q})\nabla U_{i,q}-(U_i-U_{i,q})(f(w_i+U_{i,q})-f(w_i)))\,dx$$
$$\equiv I_1+I_2+I_3.$$

By the Taylor expansion of f around w_i, we have

$$f(w_i+\xi)-f(w_i)=f'(w_i)\xi+\int_0^1(1-\eta)f''(w_i+\eta\xi)\,d\eta\cdot\xi^2,$$

$$I_1\geqslant\frac{1}{2}\int_{D_i}(|\nabla U_{i,q}|^2-f'(w_i)U_{i,q}^2)\,dx$$
$$-\frac{1}{3}\sup_{\xi\in\mathbb{R}}|f''(\xi)|\int_{D_i}|U_{i,q}|^3\,dx$$
$$\geqslant\frac{1}{2}\mu_1(w_i,D_i)\|U_{i,q}\|_{L^2(D_i)}^2-\frac{1}{3}c_2\|U_{i,q}\|_{L^3(D_i)}^3,$$

$$
\begin{aligned}
I_3 &\geqslant \int_{D_i} (\nabla(U_i - U_{i,q}) \nabla U_{i,q} - f'(w_i)(U_i - U_{i,q}) U_{i,q}\, dx \\
&\quad - c_2 \int_{D_i} |U_i - U_{i,q}| \cdot |U_{i,q}|^2\, dx \\
&\geqslant -c_2 \|U_i - U_{i,q}\|_{L^2(D_i)} \cdot \|U_{i,q}\|_{L^4(D_i)}^2, \\
I_2 &\geqslant \frac{1}{2} \int_{D_i} (|\nabla(U_i - U_{i,q})|^2 - c_1 |U_i - U_{i,q}|^2)\, dx \\
&= \frac{1}{2} \int_{D_i} (|\nabla(U_i - U_{i,q})|^2 - f'(w_i) |U_i - U_{i,q}|^2)\, dx \\
&\quad + \frac{1}{2} \int_{D_i} (f'(w_i) - c_1) |U_i - U_{i,q}|^2\, dx \\
&\geqslant \frac{1}{2} (\mu_{q+1}(w_i, D_i) - 2c_1) \|U_i - U_{i,q}\|_{L^2(D_i)}^2.
\end{aligned}
$$

Here we have put $c_1 \equiv \sup_{\xi \in \mathbb{R}} |f'(\xi)|$, $c_2 \equiv \sup_{\xi \in \mathbb{R}} |f''(\xi)|$.

First we take q sufficiently large so that, $\mu_{q+1}(w_i, D_i) - 2c_1 \geqslant 2$ for any i $(1 \leqslant i \leqslant N)$ and fix this q. Let $\mathscr{E}_{i,q}$ be the linear subspace of $L^2(D_i)$ generated by $\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,q}$, i.e., $\mathscr{E}_{i,q} \equiv$ L.h.$[\varphi_{i,1}, \varphi_{i,2}, ..., \varphi_{i,q}]$. As $\mathscr{E}_{i,q}$ is a finite dimensional norm space, any two norms on $\mathscr{E}_{i,q}$ are equivalent to each other. Therefore we have the following inequalities. There exists a positive constant $\gamma_q > 0$ such that

$$
\begin{aligned}
&\gamma_q \leqslant \|\varphi\|_{L^3(D_i)} / \|\varphi\|_{L^2(D_i)} \leqslant 1/\gamma_q \qquad \text{for any} \quad \varphi \in \mathscr{E}_{i,q}, \\
&\gamma_q \leqslant \|\varphi\|_{L^4(D_i)} / \|\varphi\|_{L^2(D_i)} \leqslant 1/\gamma_q \qquad \text{for any} \quad \varphi \in \mathscr{E}_{i,q}.
\end{aligned} \tag{2.7}
$$

We remark that $U_{i,q} \in \mathscr{E}_{i,q}$.

Therefore we have

$$
\begin{aligned}
&\int_{D_i} \left(\frac{1}{2} |\nabla U|^2 - \int_A^{A+U} (f(\xi) - f(A))\, d\xi \right) dx \\
&\quad \geqslant \frac{\mu_1(w_i, D_i)}{2} \|U_{i,q}\|_{L^2(D_i)}^2 - \frac{c_2}{3\gamma_q^3} \|U_{i,q}\|_{L^2(D_i)}^3 + \|U_i - U_{i,q}\|_{L^2(D_i)}^2 \\
&\qquad - \frac{c_2}{\gamma_q^2} \|U_i - U_{i,q}\|_{L^2(D_i)} \cdot \|U_{i,q}\|_{L^2(D_i)}^2 \\
&\quad \geqslant \left(\frac{\mu_1(w_i, D_i)}{2} - \frac{c_2}{3\gamma_q^3} \|U_{i,q}\|_{L^2(D_i)} - \frac{c_2^2}{2\gamma_q^4} \|U_{i,q}\|_{L^2(D_i)}^2 \right) \|U_{i,q}\|_{L^2(D_i)}^2 \\
&\qquad + \frac{1}{2} \|U_i - U_{i,q}\|_{L^2(D_i)}^2.
\end{aligned} \tag{2.8}
$$

Let $h_i(\xi) \equiv (\frac{1}{2}\mu_1(w_i, D_i) - (c_2/3\gamma_q^3)\xi - (c_2^2/2\gamma_q^4)\xi^2)$, $\xi \geqslant 0$ $(1 \leqslant i \leqslant N)$. There exists a constant $\delta_1 > 0$, such that $h_i(\xi) \geqslant (\mu_1(w_i, D_i))/4$ for any $\xi \in [0, \delta_1]$ $(1 \leqslant i \leqslant N)$ because $\mu_1(w_i, D_i) > 0$. Therefore if $\|U\|_{L^2(D_i)} \leqslant \delta_1$, $\|U_{i,q}\|_{L^2(D_i)} \leqslant \delta_1$ holds and so by the inequalities (2.3) and (2.8), we have

$$T_\zeta(U + A) \geqslant \sum_{i=1}^{N} \min\left(\frac{1}{2}, \frac{\mu_1(w_i, D_i)}{4}\right) \|U\|^2_{L^2(D_i)} - \int_{Q(\zeta)} \int_A^{A+U} f(\xi)\, d\xi\, dx.$$

By this inequality we complete the proof of Lemma 2.2.

By Lemma 2.2, we will estimate $\|u_\zeta(t, \cdot) - A(\cdot)\|_{L^2(D_i)}$ $(1 \leqslant i \leqslant N)$. We put

$$c_3 \equiv \sup_{x \in \mathbb{R}^n} |\nabla A|^2 + \int_{\underline{M}-2}^{\bar{M}+2} |f(\xi)|\, d\xi.$$

Define a positive function $\zeta(\delta) > 0$ in some interval $(0, \delta_2)$ by

$$\frac{1}{4} \min_{1 \leqslant i \leqslant N} \min \left\{\frac{1}{2}, \frac{\mu_1(w_i, D_i)}{2}\right\} \delta^2 - \delta^3 = c_3 \operatorname{Vol}(Q(\zeta(\delta))).$$

By retaking $\delta_2 > 0$, smaller if necessary, $\zeta(\delta)$ is monotone in the interval $(0, \delta_2)$ and $0 < \delta_2 < \delta_1$. $\lim_{\delta \to 0} \zeta(\delta) = 0$ also holds. Let $\delta(\zeta)$ be the inverse function of $\zeta(\delta)$ which is defined in $(0, \zeta_1]$ $(\zeta_1 = \zeta(\delta_2/3))$. Then $\lim_{\zeta \to 0} \delta(\zeta) = 0$ holds. Here we have the following result.

LEMMA 2.3. *For any* $\zeta \in (0, \zeta_1]$ *and any* $B \in E(\delta, \zeta)$ *where* $\delta(\zeta) \leqslant \delta \leqslant 2\delta(\zeta)$, *then* $\|u_\zeta(t, \cdot) - A(\cdot)\|_{L^2(D_i)} \leqslant \delta$ $(0 \leqslant t < \infty)$ *for* $i = 1, 2, ..., N$ *where* u_ζ *is the solution of* (2.4).

Proof of Lemma 2.3. $\zeta \in (0, \zeta_1]$, $\delta(\zeta) \leqslant \delta \leqslant 2\delta(\zeta)$. By $2\delta(\zeta) \leqslant 2\delta_2/3 \leqslant 2\delta_1/3$, $\|B - A\|_{L^2(D_i)} \leqslant 2\delta_1/3 < \delta_1$, there exists a maximal t_1 $(0 < t_1 \leqslant \infty]$ such that $\|u_\zeta(t, \cdot) - A(\cdot)\|_{L^2(D_i)} \leqslant \delta_1$ for any $t \in [0, t_1]$ and any $i = 1, ..., N$. By applying Lemma 2.2, we have, for any $t \in [0, t_1]$,

$$\begin{aligned}
\min_{1 \leqslant i \leqslant N} \min &\left\{\frac{1}{2}, \frac{\mu_1(w_i, D_i)}{2}\right\} \sum_{i=1}^{N} \|u_\zeta(t, \cdot) - A(\cdot)\|^2_{L^2(D_i)} \\
&\leqslant T_\zeta(u_\zeta(t, \cdot)) + \int_{\underline{M}-2}^{\bar{M}+2} |f(\xi)|\, d\xi \cdot \operatorname{Vol}(Q(\zeta)) \\
&\leqslant T_\zeta(A) + \delta^3 + \int_{\underline{M}-2}^{\bar{M}+2} |f(\xi)|\, d\xi \cdot \operatorname{Vol}(Q(\zeta)) \\
&\leqslant c_3 \operatorname{Vol}(Q(\zeta)) + \delta^3 \leqslant c_3 \operatorname{Vol}(Q(\zeta(\delta))) + \delta^3 \\
&= \frac{1}{4} \min_{1 \leqslant i \leqslant N} \left\{\frac{1}{2}, \frac{\mu_1(w_i, D_i)}{2}\right\} \delta^2 \qquad (0 \leqslant t \leqslant t_1). \qquad (2.9)
\end{aligned}$$

If t_1 is finite, we have by the above inequality

$$\|u_\zeta(t, \cdot) - A(\cdot)\|_{L^2(D_i)} \leqslant \delta/2 \leqslant \delta_1/3 \text{ holds for } t \in [0, t_1].$$

By the continuity of $\|u_\zeta(t, \cdot) - A\|_{L^2(D_i)}$ in t, there exists $t' > t_1$ such that $\|u_\zeta(t, \cdot) - A\|_{L^2(D_i)} \leqslant \delta_1$ $(0 \leqslant t \leqslant t')$. But this contradicts the maximality of t_1. Therefore $t_1 = \infty$. Again by (2.9) for $t_1 = \infty$, we have the inequality $\|u_\zeta(t, \cdot) - A(\cdot)\|_{L^2(D_i)} \leqslant \delta$, for $0 \leqslant t < \infty$.

Thus we have constructed a family of the closed invariant subsets $E(\delta, \zeta)$ $(\delta(\zeta) \leqslant \delta \leqslant 2\delta(\zeta))$ for $\zeta \in (0, \zeta_1]$, and applying [9, Theorem 4.2], we obtain a stable solution v_ζ in $E(\delta(\zeta), \zeta)$ for $\zeta \in (0, \zeta_1]$. The first line of (2.2) is a direct result of

$$\|v_\zeta - w_i\|_{L^2(D_i)} \leqslant \delta(\zeta), \qquad \lim_{\zeta \to 0} \delta(\zeta) = 0. \tag{2.10}$$

The second line of (2.2) can be obtained by the bootstrap method with the elliptic regularity theorem with (2.10). Thus we have completed the proof of Theorem 1.

3. Characterization Theorem

In this section, we present the main result of this paper. We deal with a singularly perturbed domain $\Omega(\zeta)$ such as in Fig. 1 where some part $Q(\zeta)$ of the domain $\Omega(\zeta)$ degenerates into a one-dimensional set (a segment) when the parameter $\zeta \to 0$. We will investigate the behavior of the solution of the following semilinear elliptic boundary value problem (3.1) and that of its linearized first eigenvalue (which usually characterize the stability of the solution). We will relate the solution for small $\zeta > 0$ and its linearized first eigenvalue to some solutions and their eigenvalues in the limit set $\lim_{\zeta \to 0} \Omega(\zeta)$ where $\Omega(\zeta)$ and f are to be defined as below in (III.1), (III.2), and (III.3).

$$\begin{aligned} \Delta v + f(v) &= 0 \qquad \text{in} \quad \Omega(\zeta), \\ \frac{\partial v}{\partial \nu} &= 0 \qquad \text{on} \quad \partial\Omega(\zeta). \end{aligned} \tag{3.1}$$

We set the domain $\Omega(\zeta)$ in the form

$$\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta),$$

where D_i $(i = 1, 2)$ and $Q(\zeta)$ are defined in (III.1) and (III.2) below where $x' = (x_2, x_3, ..., x_n) \in \mathbb{R}^{n-1}$.

(III.1) D_1 and D_2 are bounded domains in $\mathbb{R}^n$ where $\bar{D}_1 \cap \bar{D}_2 = \varnothing$ and each D_i has a smooth boundary ∂D_i and the following conditions hold for some positive constant $\zeta_* > 0$:

$$\bar{D}_1 \cap \{x = (x_1, x') \in \mathbb{R}^n \mid x_1 \leqslant 1, |x'| < 3\zeta_*\}$$
$$= \{(1, x') \in \mathbb{R}^n \mid |x'| < 3\zeta_*\},$$
$$\bar{D}_2 \cap \{x = (x_1, x') \in \mathbb{R}^n \mid x_1 \geqslant -1, |x'| < 3\zeta_*\}$$
$$= \{(-1, x') \in \mathbb{R}^n \mid |x'| < 3\zeta_*\}.$$

(III.2) $Q(\zeta) = R_1(\zeta) \cup R_2(\zeta) \cup \Gamma(\zeta)$ where

$$R_1(\zeta) = \{(x_1, x') \in \mathbb{R}^n \mid 1 - 2\zeta < x_1 \leqslant 1, |x'| < \zeta\rho((x_1 - 1)/\zeta)\},$$
$$R_2(\zeta) = \{(x_1, x') \in \mathbb{R}^n \mid -1 \leqslant x_1 < -1 + 2\zeta, |x'| < \zeta\rho((-1 - x_1)/\zeta)\},$$
$$\Gamma(\zeta) = \{(x_1, x') \in \mathbb{R}^n \mid -1 + 2\zeta \leqslant x_1 \leqslant 1 - 2\zeta, |x'| < \zeta\},$$

where $\rho \in C^0((-2, 0]) \cap C^\infty((-2, 0))$ is a positive function such that $\rho(0) = 2$, $\rho(s) = 1$ for $s \in (-2, -1)$, $d\rho/ds > 0$ for $s \in (-1, 0)$, and the inverse function $\rho^{-1}: (1, 2) \to (-1, 0)$ saisfies $\lim_{\xi \uparrow 2-0} d^k\rho^{-1}/d\xi^k = 0$ holds for any positive integer $k \geqslant 1$. We put the points $p_1 = (1, 0, ..., 0)$ and $p_2 = (-1, 0, ..., 0)$.

We impose the following condition on the nonlinear terms f:

(III.3) $f \in C^\infty(\mathbb{R})$, $\overline{\lim}_{\xi \to \infty} f(\xi) < 0$, $\underline{\lim}_{\xi \to -\infty} f(\xi) > 0$.

We assume the above conditions (III.1), (III.2), and (III.3) in this section and also in Sections 5, 6, 7. Our main results are stated in the following theorems. The asymptotic behavior of the solution when $\zeta \to 0$ is characterized in Theorem 2 and that of the linearized first eigenvalue of the solution is characterized in Theorem 3.

THEOREM 2. *Assume $n \geqslant 3$. For any $\zeta \in (0, \zeta_*)$, let v_ζ be an arbitrary solution of* (3.1). *Then, for any sequence of positive values $\{\zeta_m\}_{m=1}^\infty$ such that $\lim_{m \to \infty} \zeta_m = 0$, there exists a subsequence $\{\sigma_m\}_{m=1}^\infty \subset \{\zeta_m\}_{m=1}^\infty$ and functions $w_1 \in C^\infty(\bar{D}_1)$, $w_2 \in C^\infty(\bar{D}_2)$, $V \in C^\infty([-1, 1])$ such that Eqs.* (3.2), (3.3) *and the uniform convergence properties* (3.4), (3.5) *hold.*

$$\begin{aligned} \Delta w_i + f(w_i) &= 0 \qquad in \quad D_i, \\ \frac{\partial w_i}{\partial \nu} &= 0 \qquad on \quad \partial D_i\ (i = 1, 2), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\frac{d^2}{dz^2} V + f(V) = 0 \qquad for \quad z \in (-1, 1), \\ &V(1) = w_1(p_1), \qquad V(-1) = w_2(p_2), \end{aligned} \tag{3.3}$$

$$\lim_{m \to \infty} \sup_{x \in D_i} |v_{\sigma_m}(x) - w_i(x)| = 0 \qquad (i = 1, 2), \tag{3.4}$$

$$\lim_{m \to \infty} \sup_{x = (x_1, x') \in Q(\sigma_m)} |v_{\sigma_m}(x_1, x') - V(x_1)| = 0. \tag{3.5}$$

Next we state the result concerning the stability of the solution v_ζ, i.e., we describe the linearized first eigenvalue of the solution.

Notation 2. We denote by $\{\lambda_k(V)\}_{k=1}^{\infty}$ and $\{\Phi_k(V)\}_{k=1}^{\infty}$, respectively, the eigenvalues (arranged in increasing order) and the orthonormalized complete system of the eigenvalues of the following eigenvalue problem:

$$\begin{gathered} \frac{d^2}{dz^2} \Phi + f'(V)\, \Phi + \lambda \Phi = 0, \qquad -1 < z < 1, \\ \Phi(-1) = \Phi(1) = 0. \end{gathered} \tag{3.6}$$

Now we have the following asymptotic behavior of the linearized first eigenvalue of the solution v_{σ_m} in Theorem 2.

THEOREM 3.

$$\lim_{m \to \infty} \mu_1(v_{\sigma_m}, \Omega(\sigma_m)) = \min\{\lambda_1(V), \mu_1(w_1, D_1), \mu_1(w_2, D_2)\}.$$

Remark. In Theorems 2 and 3, we assumed that the space dimension $n \geqslant 3$ for the technical reason concerning the method of the proof. We are not sure whether this assumption is essential or not.

Remark. Theorems 2 and 3 are generalizations and elaborations of Theorems 2 and 3 in [8].

The proofs of the above theorems are given in Sections 5 and 6.

4. COMPARISON FUNCTIONS

In the proof of the main results, we will use a comparison function to prove the uniform convergence of the solutions $D_1 \cup D_2$, especially in the neighborhood of the points p_1 and p_2. This comparison function is a linear combination of the two radially symmetric solutions of the Helmholtz equation,

$$\Delta \phi + M\phi = 0, \tag{4.1}$$

where M is a positive constant (which is to be chosen suitably large in Sections 5 and 6). The radially symmetric solution ϕ of (4.1) satisfies:

$$\frac{d^2}{dr^2} \phi + \frac{n-1}{r} \frac{d}{dr} \phi + M\phi = 0, \qquad r \in (0, \infty). \tag{4.2}$$

The two linearly independent solutions of (4.2) are expressed as

(i) $n \geqslant 3$, odd,

$$\phi_1(r) = (M^{1/2}r)^{-(n-2)/2} J_{(n-2)/2}(M^{1/2}r),$$
$$\phi_2(r) = (M^{1/2}r)^{-(n-2)/2} J_{-(n-2)/2}(M^{1/2}r),$$

(ii) $n \geqslant 2$, even,

$$\phi_1(r) = (M^{1/2}r)^{-(n-2)/2} J_{(n-2)/2}(M^{1/2}r)$$
$$\phi_2(r) = -(M^{1/2}r)^{-(n-2)/2} Y_{(n-2)/2}(M^{1/2}r).$$

Here J_ν and Y_N are respectively the Bessel function and the Neumann function, i.e.,

$$J_\nu(r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(\nu+m+1)\,\Gamma(m+1)} (r/2)^{\nu+2m},$$
$$Y_N(r) = \frac{2}{\pi} J_N(r) \log\frac{r}{2} - \frac{1}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{\psi(m+1)+\psi(N+m+1)}{m!\,(N+m)!} (r/2)^{N+2m}$$
$$-\frac{1}{\pi} \sum_{m=0}^{N-1} \frac{(N-1-m)!}{m!} (r/2)^{-N+2m}$$

(see K. Yosida [14, pp. 51–55]).

We use the following properties of ϕ_1 and ϕ_2.

LEMMA 4.1. *Assume $n \geqslant 3$. For any positive constant M, there exist positive constants $c_0(M)$, $c_1(M)$, and $c_2(M)$ such that ϕ_1 and ϕ_2 are expressed as*

$$\phi_1(r) = A_1(r), \qquad \phi_2(r) = \frac{A_2(r)}{r^{n-2}},$$

where $A_1 \in C^\infty([0, \infty))$ and $A_2 \in C^\infty((0, \infty))$ satisfy the properties

$$0 < c_1(M) \leqslant A_1(r) \leqslant c_2(M) \qquad \text{for} \quad r \in [0, c_0(M)],$$
$$0 < c_1(M) \leqslant A_2(r) \leqslant c_2(M) \qquad \text{for} \quad r \in (0, c_0(M)].$$

5. PROOF OF THEOREM 2

Put $\bar{M} \equiv \sup\{\xi \in \mathbb{R} \mid f(\xi) = 0\}$, $\underline{M} \equiv \inf\{\xi \in \mathbb{R} \mid f(\xi) = 0\}$. It is easy to see that

$$\underline{M} \leqslant v_\zeta(x) \leqslant \bar{M} \qquad \text{for} \quad x \in \Omega(\zeta) \ (0 < \zeta \leqslant \zeta_*). \tag{5.1}$$

By integrating Eq. (3.1) on $\Omega(\zeta)$ after multiplying by v_ζ, we get the following inequality with the aid of (5.1) for some constant $c>0$:

$$\int_{\Omega(\zeta)} |\nabla v_\zeta|^2 \, dx = \int_{\Omega(\zeta)} v_\zeta f(v_\zeta) \, dx \leqslant c < +\infty \qquad (0<\zeta\leqslant\zeta_*). \tag{5.2}$$

By the Rellich's theorem, there exist a subsequence $\{v_{\zeta_{m'}}\} \subset \{v_{\zeta_m}\}$ and functions $w_i \in L^2(D_i)$ $(i=1, 2)$ such that

$$\lim_{m \to \infty} \|v_{\zeta_{m'}} - w_i\|_{L^2(D_i)} = 0 \qquad (i=1, 2). \tag{5.3}$$

Furthermore, by the bootstrap argument concerning the convergence of v_ζ, we conclude that $w_i \in C^\infty(\bar{D}_i - \{p_i\})$ and w_i satisfies the equation in D_i and the Neumann boundary condition on $\partial D_i - \{p_i\}$. By applying Proposition 8.1 to w_i in D_i, we conclude that p_i is a removable singularity and w_i satisfies the Neumann boundary condition up to ∂D_i and we have the following for each $i=1, 2$,

$$\begin{aligned} \Delta w_i + f(w_i) &= 0 \qquad \text{in} \quad D_i, \\ \partial w_i/\partial \nu &= 0 \qquad \text{on} \quad \partial D_i. \end{aligned} \tag{5.4}$$

$$\lim_{m \to \infty} v_{\zeta_{m'}} = w_i \ \text{ in } \ C^\infty(\overline{D_i - \Sigma_i(\eta)}) \ \text{ for any } \ \eta > 0, \tag{5.5}$$

where

$$\Sigma_1(\eta) \equiv \{(x_1, x') \in \mathbb{R}^n \,|\, x_1 > 1, |x - p_1| < \eta\},$$

$$\Sigma_2(\eta) \equiv \{(x_1, x') \in \mathbb{R}^n \,|\, x_1 < -1, |x - p_2| < \eta\}.$$

Hereafter we will prove

$$\lim_{m \to \infty} \sup_{x \in D_i} |v_{\zeta_{m'}}(x) - w_i(x)| = 0 \qquad (i=1, 2). \tag{5.6}$$

Without loss of generality, we deal with only the case $i=1$ and we introduce the following set and the parameter:

$$K(\varepsilon, \zeta) \equiv \{x \in D_1 \,|\, |v_\zeta(x) - w_1(x)| \geqslant \varepsilon\},$$

$$\eta(\varepsilon, \zeta) \equiv \inf\{\eta > 0 \,|\, K(\varepsilon, \zeta) \subset \Sigma_1(\eta)\}.$$

Suppose that (5.6) does not hold for $i=1$, there exist a positive constant $\varepsilon_0 > 0$ and a subsequence $\{\kappa_m\} \subset \{\zeta_{m'}\}$ such that,

$$\eta(\varepsilon_0, \kappa_m) > 0 \qquad \text{for any} \quad m \geqslant 1.$$

It is easy to see $\lim_{m \to \infty} \eta(\varepsilon_0, \kappa_m) = 0$ by (5.5).

Under the above assumption, we will deduce a contradiction. We separate the argument into the following two cases.

Case 1. $\lim_{m\to\infty} \kappa_m/\eta(\varepsilon_0, \kappa_m) = 0$,

Case 2. $\lim_{m\to\infty} \kappa_m/\eta(\varepsilon_0, \kappa_m) > 0$.

We prepare an auxiliary inequality which is very important in this proof. Put $M \equiv 2\max_{\underline{M}\leqslant\xi\leqslant\bar{M}} |f'(\xi)| + 1$ and $c_*(M) \equiv \min\{3\zeta_*, c_0(M)\}$ where $c_0(M)$ is the parameter introduced in Section 4 for the above M. We also put

$$\alpha_m \equiv \sup_{x\in D_1,\, |x-p_1| = c_*(M)} |v_{\kappa_m}(x) - w_1(x)| + \frac{1}{m} \quad \text{and} \quad \gamma_m \equiv \max\{2\kappa_m, \eta_m\},$$

where $\eta_m \equiv \eta(\varepsilon_0, \kappa_m)$.

It is easy to see $\lim_{m\to\infty} \alpha_m = 0$ by (5.6).

LEMMA 5.1.

$$|v_{\kappa_m}(x) - w_1(x)| < \frac{\alpha_m}{c_1(M)} A_1(|x-p_1|) + \frac{\varepsilon_0}{c_1(M)} \frac{\gamma_m^{n-2} A_2(|x-p_1|)}{|x-p_1|^{n-2}},$$

for $x \in \Sigma_1(c_*(M)) - \Sigma_1(\gamma_m)$.

Proof of Lemma 5.1. We denote by ϕ the right hand side of the inequality. Then by Lemma 4.1 and the definition α_m and γ_m, the inequality holds for $x \in \bar{D}_1$ such that $|x-p_1| = c_*(M)$ or $|x-p_1| = \gamma_m$. Let $h^* \equiv \sup\{h \in [0, 1] \mid \phi(x) - h(v_{\kappa_m} - w_1) \geqslant 0$ in $\Sigma_1(c_*(M)) - \Sigma_1(\gamma_m)\}$. If $h^* < 1$, there exists $x^* \in \bar{D}_1$ such that

$$\begin{aligned} &\gamma_m < |x^*-p_1| < c_*(M) \qquad \text{and} \qquad \phi(x^*) - h^*(v_{\kappa_m}(x^*) - w_1(x^*)) = 0, \\ &\phi(x) - h^*(v_{\kappa_m}(x) - w_1(x)) \geqslant 0 \qquad \text{for any} \quad x \in \Sigma_1(c_*(M)) - \Sigma_1(\gamma_m). \end{aligned} \tag{5.7}$$

But we can deduce the following differential inequality from (3.1) and (4.1):

$$\begin{aligned} &\Delta(\phi - h^*(v_{\kappa_m} - w_1)) + \left(\int_0^1 f'(sv_{\kappa_m} + (1-s)w_1)\, ds\right)(\phi - h^*(v_{\kappa_m} - w_1)) \\ &\quad = \left(\int_0^1 f'(sv_{\kappa_m} + (1-s)w_1)\, ds - M\right)\phi \leqslant -\phi < 0 \text{ in } \Sigma_1(c_*(M)) - \Sigma_1(\gamma_m), \end{aligned} \tag{5.8}$$

$$\frac{\partial}{\partial \nu}(\phi - h^*(v_{\kappa_m} - w_1)) = 0 \qquad \text{on} \quad \partial D_1 \cap \{\gamma_m \leqslant |x-p_1| \leqslant c_*(M)\}. \tag{5.9}$$

Therefore if x^* is an interior point of $\Sigma_1(c_*(M))-\Sigma_1(\gamma_m)$, the above differential inequality (5.8) contradicts the existence of such a point x^* by the Strong Maximum Principle.

On the other hand, if $x^*\in\partial D_1\cap\{\gamma_m<|x-p_1|<c_*(M)\}$, again the differential inequality (5.8) and the Neumann boundary condition (5.9) contradict the existence of such a point x^* by the Hopf Lemma. This concludes $h^*=1$. By the same argument as above for $h^*=1$ in (5.8) and (5.9), we obtain $\phi(x)-(v_{\kappa_m}(x)-w_1(x))>0$ for any $x\in\overline{\Sigma_1(c_*(M))-\Sigma_1(\gamma_m)}$.

By an argument similar to

$$\begin{aligned}h_*=\sup\{h\in[0,1]\mid \phi(x)-h(w_1(x)-v_{\kappa_m}(x))\geqslant 0\\ \text{in}\quad \Sigma_1(c_*(M))-\Sigma_1(\gamma_m)\}\end{aligned}$$

we obtain the conclusion of Lemma 5.1.

Case 1. By taking the subsequence of $\{\kappa_m\}$ if necessary, we assume without loss of generality that $\lim_{m\to\infty}\kappa_m/\eta_m=0$ in Case 1. Remark that $\gamma_m=\eta_m$ for large m in this case. We change the scale of the variable as

$$\begin{aligned}&x-p_1=\gamma_m(y-p_1),\\ &U_m(y)=v_{\kappa_m}(p_1+\gamma_m(y-p_1)),\end{aligned}\tag{5.10}$$

$$\begin{aligned}&\Delta_y U_m+\gamma_m^2 f(U_m)=0, &&\text{for}\quad y\in\Sigma_1(c_*(M)/\gamma_m),\\ &\frac{\partial U_m}{\partial y_1}(0,y')=0, &&2\kappa_m/\gamma_m\leqslant|y'|\leqslant c_*(M)/\gamma_m.\end{aligned}\tag{5.11}$$

By using the definition of η_m, we have

$$\sup_{y_1>1,|y|=1}|U_m(y)-w_1(p_1+\gamma_m(y-p_1))|=\varepsilon_0\qquad\text{for large } m. \tag{5.12}$$

On the other hand, we have the following estimate by Lemma 5.1:

$$\begin{aligned}&|U_m(y)-w_1(p_1+\gamma_m(y-p_1))|\\ &\quad<\frac{\alpha_m}{c_1(M)}A_1(\gamma_m|y-p_1|)+\frac{\varepsilon_0}{c_1(M)}\frac{A_2(\gamma_m|y-p_1|)}{|y-p_1|^{n-2}}.\end{aligned}$$

By the bootstrap method and the Schauder estimate, $\{U_m\}$ is compact in $C^\infty(\{(y_1,y')\in\mathbb{R}^n\mid y_1\geqslant 1,\ \eta\leqslant|y-p_1|\leqslant 1/\eta\})$ for any $\eta>0$. Remark $\lim_{m\to\infty}\kappa_m/\eta_m=0$. Therefore passing to some subsequence in (5.11), we get a function $C^\infty(\{(y_1,y')\mid y_1\geqslant 1, y\neq p_1\})$ which satisfies the properties

$$\begin{aligned}
&\Delta_y U=0 && \text{in} \quad \{(y_1, y')\in\mathbb{R}^n \mid y_1>1\},\\
&\frac{\partial U}{\partial y_1}(0, y')=0 && \text{for} \quad y'\in\mathbb{R}^{n-1}-\{0\}, \qquad (5.13)\\
&\underline{M}\leqslant U(y)\leqslant \bar{M} && \text{in} \quad \{(y_1, y')\in\mathbb{R}^n \mid y_1>1\}.
\end{aligned}$$

$$|U(y)-w_1(p_1)|\leqslant\frac{c_2(M)\varepsilon_0}{c_1(M)|y-p_1|^{n-2}} \quad \text{in } \{(y_1, y')\mid y_1\geqslant 1, |y-p_1|\leqslant 1\},$$

$$\sup_{y_1\geqslant 1, |y-p_1|=1} |U(y)-w_1(p_1)|=\varepsilon_0. \qquad (5.14)$$

By reflecting the function about the hyperplane $y_1=1$ by the aid of the Neumann boundary condition, we get a bounded harmonic function in $\mathbb{R}^n-\{p_1\}$. Furthermore, applying the removable singularity theorem, we obtain a bounded harmonic function $\bar{U}$ on $\mathbb{R}^n$ with the properties

$$\sup_{|y-p_1|=1} |\bar{U}(y)-w_1(p_1)|=\varepsilon_0>0, \qquad \lim_{|y|\to\infty} \bar{U}(y)=w_1(p_1). \qquad (5.15)$$

The existence of such a function contradicts the Harnack inequality. We complete the proof of (5.6) in Case 1.

Case 2. Take a constant $\beta>0$ such that

$$\min\left\{\frac{1}{2}, \lim_{m\to\infty} \kappa_m/\eta(\varepsilon_0, \kappa_m)\right\}>\beta>0.$$

We change the scale of the variable around the point p_1 as

$$\begin{aligned}
&x-p_1=\kappa_m(y-p_1), \qquad (5.16)\\
&U_m(y)=v_{\kappa_m}(p_1+\kappa_m(y-p_1)).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\Delta_y U_m+\kappa_m^2 f(U_m)=0 && \text{in} \quad H_{\kappa_m},\\
&\frac{\partial U_m}{\partial \nu}(0, y')=0 && \text{for } y' \text{ such that } 2\leqslant |y'|\leqslant c_*(M)/\kappa_m,
\end{aligned} \qquad (5.17)$$

where

$$\begin{aligned}
H_\eta\equiv &\{(y_1, y')\in\mathbb{R}^n\mid y_1>1, |y-p_1|\leqslant c_*(M)/\eta\}\\
&\cup\{(y_1, y')\in\mathbb{R}^n\mid -1\leqslant y_1\leqslant 1, |y'|<\rho(y_1-1)\}\\
&\cup\{(y_1, y')\in\mathbb{R}^n\mid 2-2/\eta\leqslant y_1\leqslant -1, |y'|<1\}.
\end{aligned}$$

and

$$H \equiv \bigcup_{\eta > 0} H_\eta.$$

We also have the following properties which are deduced from the definition of Case 2, Lemma 5.1, and $\overline{\lim}_{m \to \infty} \max(2, \eta_m/\kappa_m) < 1/\beta$:

$$\sup_{y_1 \geqslant 1, |y - p_1| \leqslant 1/\beta} |U_m(y) - w_1(p_1 + \kappa_m(y - p_1))| \geqslant \varepsilon_0 > 0,$$

$$\begin{aligned} &|U_m(y) - w_1(p_1 + \kappa_m(y - p_1))| \\ &\quad \leqslant \frac{\alpha_m A_1(\kappa_m |y - p_1|}{c_1(M)} + \frac{\varepsilon_0 A_2(\kappa_m |y - p_1|)}{c_1(M) |y - p_1|^{n-2}} \\ &\quad \text{in } \{(y_1, y') \in \mathbb{R}^n \mid y_1 \geqslant 1,\ 1/\beta \leqslant |y - p_1| \leqslant c_*(M)/\kappa_m\}. \end{aligned} \tag{5.18}$$

By almost the same argument as that in Case 1 concerning the estimate of the family of the function, we see that $\{U_m\}$ is compact in $C^\infty(\bar{H}_\eta)$ for any $\eta \in (0, 1]$. Therefore, taking the limit in (5.13) along some subsequence, we get a function $U \in C^\infty(\bar{H})$ which satisfies the properties

$$\begin{gathered} \Delta_y U = 0 \text{ in } H, \qquad \partial U/\partial \nu = 0 \text{ on } \partial H, \\ \underline{M} \leqslant U(y) \leqslant \bar{M} \text{ in } H. \end{gathered} \tag{5.19}$$

Furthermore we have, by (5.18),

$$\sup_{y_1 \geqslant 1, |y - p_1| \leqslant 1/\beta} |U(y) - w_1(p_1)| \geqslant \varepsilon_0 > 0,$$

$$\begin{aligned} &|U(y) - w_1(p_1)| \leqslant \frac{c_2(M)\varepsilon_0}{c_1(M)} \frac{1}{|y - p_1|^{n-2}} \\ &\qquad \text{in } \{(y_1, y') \in \mathbb{R}^n \mid y_1 \geqslant 1,\ 1/\beta \leqslant |y - p_1|\}. \end{aligned} \tag{5.20}$$

Thus we have obtained a bounded harmonic function in H with the Neumann boundary condition such that $\lim_{y_1 \geqslant 1, |y| \to \infty} U(y) = w_1(p_1)$. But this is impossible from the following proposition. Therefore we have completed Case 2.

PROPOSITION 5.1 (Lemma 3.2 [8]). *Let W be a bounded function in $\bar{H}$ with the following properties* (5.21):

$$\begin{gathered} \Delta_y W = 0 \text{ in } H, \qquad \partial W/\partial \nu = 0 \text{ on } \partial H, \\ \lim_{y_1 \geqslant 1, |y| \to \infty} W(y) = c. \end{gathered} \tag{5.21}$$

Then, $W \equiv c$ in H.

Therefore we have completed the proof of (5.6). Next we will prove,

$$\lim_{m\to\infty} \sup_{x\in R_i(\zeta_{m'})} |v_{\zeta_{m'}}(x) - w_i(p_i)| = 0 \qquad (i=1, 2). \tag{5.22}$$

We recall the change of the scale of the variable such as (5.16). Put $W_m(y) = v_{\zeta_{m'}}(p_1 + \zeta_{m'}(y - p_1))$ and in the same manner as in Case 1, we see that $\{W_m\}$ is compact in $C^\infty(\bar{H}_\eta)$ for any $\eta > 0$. Therefore, any subsequence $\{W_{m'}\} \subset \{W_m\}$ has a convergent subsequence $\{W_{m''}\}$ of $\{W_{m'}\}$ and a function $W \in C^\infty(\bar{H})$ such that

$$\begin{aligned} &\lim_{m\to\infty} W_{m''} = W \text{ in } C^\infty(\bar{H}_\eta) \text{ for any } \eta > 0, \\ &\Delta_y W = 0 \text{ in } H, \qquad \partial W/\partial \nu = 0 \text{ on } \partial H. \end{aligned} \tag{5.23}$$

On the other hand, by (5.6), we have

$$\lim_{m\to\infty} \sup_{y\in \Sigma_1(\eta)} |W_{m''}(y) - w_1(p_1)| = 0$$

for any $\eta > 0$, and so we conclude that $W(y) = w_1(p_1)$ for any $y = (y_1, y')$ such that $y_1 \geq 1$. Applying the unique continuation theorem to the harmonic function W, we conclude $W(y) = w_1(p_1)$ for any $y \in H$. Thus we have obtained (5.22).

The rest of the proof of Theorem 2 is almost parallel to the former half of the proof of Theorem 3 in [8] because we have already established (5.22). Therefore we only mention here without proof the desired result (Proposition 5.2). We apply it to (3.1), (5.1), and (5.22) and we get a subsequence $\{\sigma_m\} \subset \{\kappa_m\}$ and a function $V \in C^2([-1, 1])$ which satisfy (3.3) and (3.5). Remark that V necessarily belongs to $C^\infty([-1, 1])$ because of (3.3). Thus we have completed the proof of Theorem 2.

PROPOSITION 5.2. *Let $\{\zeta_m\}_{m=1}^\infty$ be a sequence of positive values such that $\lim_{m\to\infty} \zeta_m = 0$ and v_{ζ_m} be an arbitrary function in $C^2(\overline{Q(\zeta_m)})$ with*

$$\begin{aligned} &\Delta v_{\zeta_m} + q_{\zeta_m}(x)\, g(v_{\zeta_m}) = 0 \qquad \text{in} \quad Q(\zeta_m), \\ &\partial v_{\zeta_m}/\partial \nu = 0 \qquad \text{on} \quad \partial Q(\zeta_m) - (\partial D_1 \cup \partial D_2), \\ &\lim_{m\to\infty} \sup_{x\in R_i(\zeta_m)} |v_{\zeta_m}(x) - d_i| = 0 \qquad (i=1, 2), \\ &\sup_{m\geq 1} \sup_{x\in Q(\zeta_m)} |v_{\zeta_m}(x)| < +\infty, \end{aligned}$$

where $d_1, d_2 \in \mathbb{R}$, $g \in C^1(\mathbb{R})$, and $q_{\zeta_m} \in C^0(\overline{Q(\zeta_m)})$ has the following asymptotic behavior for some $q \in C^0([-1, 1])$:

$$\lim_{m\to\infty} \sup_{x\in Q(\zeta_m)} |q_{\zeta_m}(x_1, x') - q(x_1)| = 0.$$

Then there exist a subsequence $\{\kappa_m\}_{m=1}^{\infty} \subset \{\zeta_m\}_{m=1}^{\infty}$ *and* $V \in C^2([-1, 1])$ *such that*

$$\frac{d^2 V}{dz^2} + q(z)\, g(V(z)) = 0, \qquad -1 < z < 1,$$

$$V(1) = d_1, \qquad V(-1) = d_2,$$

$$\lim_{m \to \infty} \sup_{x \in Q(\kappa_m)} |v_{\kappa_m}(x_1, x') - V(x_1)| = 0.$$

6. Proof of Theorem 3

We consider the asymptotic behavior of the linearized first eigenvalue $\mu_1(v_{\sigma_m}, \Omega(\sigma_m))$ (which is a simple eigenvalue) when $m \to \infty$. We first prove the following partial result.

Lemma 6.1.

$$\overline{\lim_{m \to \infty}}\ \mu_1(v_{\sigma_m}, \Omega(\sigma_m)) \leqslant \min\{\mu_1(w_1, D_1), \mu_1(w_2, D_2), \lambda_1(V)\}.$$

Proof of Lemma 6.1. We define the following test functions. Let $\psi_1, \psi_2 \in C_0^{\infty}(\mathbb{R}^n)$ be functions which satisfy

$$\psi_1(x) = \begin{cases} \varphi_{1,1}(w_1, D_1)(x) & \text{for} \quad x \in D_1, \\ 0 & \text{for} \quad x \in D_2, \end{cases}$$

$$\psi_2(x) = \begin{cases} 0 & \text{for} \quad x \in D_1, \\ \varphi_{2,1}(w_2, D_2)(x) & \text{for} \quad x \in D_2. \end{cases}$$

We define $\Psi_\zeta \in H^1(\Omega(\sigma_m))$ as

$$\Psi_\zeta(x_1, x') = \begin{cases} \Phi_1(x_1) - \delta_\zeta & \text{for} \quad x \in K(\zeta), \\ 0 & \text{for} \quad x \in \Omega(\zeta) - K(\zeta), \end{cases}$$

where $\Phi_1 = \Phi_1(V)$ is the linearized first eigenfunction in (3.6) and we assumed without loss of generality that it is positive, and we defined the parameter δ_ζ and the set $K(\zeta)$ as

$$\delta_\zeta = \max\{\Phi_1(1 - 2\zeta), \Phi_1(-1 + 2\zeta)\} > 0,$$

$$K(\zeta) = \{(x_1, x') \in \Gamma(\zeta) \mid \Phi_1(x_1) \geqslant \delta_\zeta\}.$$

We remark that $\lim_{\zeta \to 0} \delta_\zeta = 0$ by $\Phi(\pm 1) = 0$.

By the min–max theorem and the above test functions, we estimate $\mu_1(v_{\sigma_m}, \Omega(\sigma_m))$ from above:

$$\begin{aligned}
&\mu_1(v_{\sigma_m}, \Omega(\sigma_m)) \\
&\quad\leqslant \int_{\Omega(\sigma_m)} (|\nabla\psi_i|^2 - f'(v_{\sigma_m})\psi_i^2)\,dx \Big/ \int_{\Omega(\sigma_m)} \psi_i^2\,dx \\
&\quad= \left\{\int_{D_i} ((|\nabla\psi_i|^2 - f'(w_i)\psi_i^2) + (f'(w_i) - f'(v_{\sigma_m}))\psi_i^2)\,dx\right. \\
&\qquad\left. + \int_{Q(\sigma_m)} (|\nabla\psi_i|^2 - f'(v_{\sigma_m})\psi_i^2)\,dx\right\} \Big/ \left\{\int_{D_i} \psi_i^2\,dx + \int_{Q(\sigma_m)} \psi_i^2\,dx\right\} \\
&\quad\leqslant \{\mu_1(w_i, D_i) + \sup_{x\in D_i} |f'(w_i) - f'(v_{\sigma_m})| \\
&\qquad + c\,\mathrm{Vol}(Q(\sigma_m))\}/(1 + c\,\mathrm{Vol}(Q(\sigma_m)) \\
&\quad\to \mu_1(w_i, D_i) \qquad \text{as} \quad m\to\infty \quad (i = 1, 2),
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
&\mu_1(v_{\sigma_m}, \Omega(\sigma_m)) \\
&\quad\leqslant \int_{\Omega(\sigma_m)} (|\nabla\Psi_{\sigma_m}|^2 - f'(v_{\sigma_m})|\Psi_{\sigma_m}|^2)\,dx \Big/ \int_{\Omega(\sigma_m)} \Psi_{\sigma_m}^2\,dx \\
&\quad= \left\{\int_{K(\sigma_m)} \left(-\Phi_1\left(\frac{\partial^2\Phi_1}{\partial x_1^2} + f'(V)\Phi_1\right) + (f'(V) - f'(v_{\sigma_m}))\Psi_{\sigma_m}^2\right) dx\right. \\
&\qquad\left. + \delta_{\sigma_m}\int_{K(\sigma_m)} \left(\frac{\partial^2\Phi_1}{\partial x_1^2} + f'(V)(2\Phi_1(x_1) - \delta_{\sigma_m})\right) dx\right\} \Big/ \\
&\qquad \int_{K(\sigma_m)} |\Phi_1(x_1) - \delta_{\sigma_m}|^2\,dx \\
&\quad\leqslant (\lambda_1(V) + \sup_{x\in K(\sigma_m)} |f'(V(x_1)) - f'(v_{\sigma_m}(x_1, x'))|) \\
&\qquad\times \int_{K(\sigma_m)} \Phi_1(x_1)^2\,dx \Big/ \int_{K(\sigma_m)} |\Phi_1(x_1) - \delta_{\sigma_m})|^2\,dx + c\,\delta_{\sigma_m} \\
&\quad\to \lambda_1(V) \qquad (m\to\infty).
\end{aligned} \tag{6.2}$$

By letting $m\to\infty$ in (6.1) and (6.2), we complete the proof of Lemma 6.1.

The normalized eigenfunction corresponding to $\mu_1(v_\zeta, \Omega(\zeta))$ is $\varphi_1(v_\zeta, \Omega(\zeta))$ (see Notations 1, 2). We denote it by φ_ζ for brevity and assume it is positive without loss of generality:

$$\begin{gathered}\varphi_\zeta \in C^\infty(\overline{\Omega(\zeta)}), \qquad \|\varphi_\zeta\|_{L^2(\Omega(\zeta))} = 1, \\ \varphi_\zeta(x) > 0 \quad \text{for} \quad x \in \overline{\Omega(\zeta)} \ (0 < \zeta \leqslant \zeta_*),\end{gathered} \tag{6.3}$$

$$\begin{aligned}\Delta\varphi_\zeta + f'(v_\zeta)\varphi_\zeta + \mu_1(v_\zeta, \Omega(\zeta))\varphi_\zeta &= 0 \quad \text{in} \quad \Omega(\zeta), \\ \partial\varphi_\zeta/\partial\nu &= 0 \quad \text{on} \quad \partial\Omega(\zeta).\end{aligned} \tag{6.4}$$

We separate the rest of the proof into the following two cases:

Case 1. $\overline{\lim}_{m\to\infty} \|\varphi_{\sigma_m}\|_{L^2(D_1 \cup D_1)} > 0.$

Case 2. $\lim_{m\to\infty} \|\varphi_{\sigma_m}\|_{L^2(D_1 \cup D_2)} = 0.$

By the min–max principle with an appropriate test function, the value μ_* is well defined:

$$-\infty < \mu_* \equiv \underline{\lim}_{m\to\infty} \mu_1(v_{\sigma_m}, \Omega(\sigma_m)) < +\infty.$$

Case 1. By the boundedness of $\{v_{\sigma_m}|_{D_1 \cup D_2}\}_{m=1}^\infty$ in $H^1(D_1 \cup D_2)$, there exists a subsequence $\{\sigma_{m'}\} \subset \{\sigma_m\}$ and a function ψ_* in $L^2(D_1 \cup D_2)$ such that

$$\begin{gathered}\psi_*(x) \geqslant 0 \ \text{in} \ D_1 \cup D_2, \qquad \|\psi_*\|_{L^2(D_1 \cup D_2)} > 0, \\ \lim_{m\to\infty} \|\varphi_{\sigma_{m'}} - \psi_*\|_{L^2(D_1 \cup D_2)} = 0, \qquad \lim_{m\to\infty} \mu_1(v_{\sigma_{m'}}, \Omega(\sigma_{m'})) = \mu_*.\end{gathered} \tag{6.5}$$

By the compactness argument and Proposition 8.1 in the Appendix, we have

$$\begin{aligned}\Delta\psi_* + f'(w)\psi_* + \mu_*\psi_* &= 0 \quad \text{in} \quad D_1 \cup D_2, \\ \partial\psi_*/\partial\nu &= 0 \quad \text{on} \quad \partial D_1 \cup \partial D_2,\end{aligned} \tag{6.6}$$

where we have put

$$w(x) \equiv \begin{cases} w_1(x) & \text{for} \quad x \in D_1 \\ w_2(x) & \text{for} \quad x \in D_2. \end{cases}$$

By $\bar{D}_1 \cap \bar{D}_2 = \varnothing$, we have the following equality,

$$\mu_* = \mu_1(w_1, D_1) \qquad \text{or} \qquad \mu_1(w_2, D_2). \tag{6.7}$$

Case 2. By $\lim_{m\to\infty} \|\varphi_{\sigma_m}\|_{L^2(D_1 \cup D_2)} = 0$ and $\|\varphi_\zeta\|_{L^2(\Omega(\zeta))} = 1$ $(0 < \zeta \leqslant \zeta_*)$, we have

$$\lim_{m\to\infty} \|\varphi_{\sigma_m}\|_{L^\infty(\Omega(\zeta))} = \infty. \tag{6.8}$$

Put $\bar{\varphi}_\zeta(x) = \varphi_\zeta(x)/\|\varphi_\zeta\|_{L^\infty(\Omega(\zeta))}$:

$$\begin{aligned} \Delta\bar{\varphi}_{\sigma_m} + f'(v_{\sigma_m})\bar{\varphi}_{\sigma_m} + \mu_1(v_{\sigma_m}, \Omega(\sigma_m))\bar{\varphi}_{\sigma_m} &= 0 \quad \text{in} \quad \Omega(\sigma_m), \\ \partial\bar{\varphi}_{\sigma_m}/\partial\nu &= 0 \quad \text{on} \quad \partial\Omega(\sigma_m). \end{aligned} \tag{6.9}$$

By the compactness argument in (6.9), we have

$$\begin{aligned} &-1 \leqslant \bar{\varphi}_{\sigma_m}(x) \leqslant 1 && \text{for} \quad x \in \Omega(\sigma_m), \\ &\lim_{m\to\infty} \bar{\varphi}_{\sigma_m} = 0 \text{ in } C^\infty(\overline{D_i - \Sigma_i(\eta)}) && \text{for any} \quad \eta > 0. \end{aligned} \tag{6.10}$$

There exists a subsequence $\{\sigma_{m'}\} \subset \{\sigma_m\}$ such that

$$\lim_{m\to\infty} \mu_1(v_{\sigma_{m'}}, \Omega(\sigma_{m'})) = \mu_*.$$

Here we can apply almost the same argument as the former part in the proof of Theorem 2 and Proposition 5.2 to (6.9) by using (6.10) and Theorem 2, and so we can characterize the asymptotic behavior of $\bar{\varphi}_{\sigma_{m''}}$ as $m \to \infty$, i.e., there exist a subsequence $\{\kappa_m\} \subset \{\sigma_{m'}\}$ and a function $\bar{\Phi}$ in $C^\infty([-1, 1])$ such that

$$\lim_{m\to\infty} \sup_{x \in D_1 \cup D_2} |\bar{\varphi}_{\sigma_m}(x)| = 0, \tag{6.11}$$

$$\begin{aligned} &\frac{d^2}{dz^2}\bar{\Phi} + (f'(V(z)) + \mu_*)\bar{\Phi} = 0 \qquad \text{in} \quad -1 < z < 1, \\ &\bar{\Phi}(\pm 1) = 0, \qquad \bar{\Phi}(z) > 0 \ (-1 < z < 1), \qquad \max_{-1 \leqslant z \leqslant 1} \bar{\Phi}(z) = 1, \end{aligned} \tag{6.12}$$

$$\lim_{m\to\infty} \sup_{x \in Q(\kappa_m)} |\bar{\varphi}_{\kappa_m}(x_1, x') - \bar{\Phi}(x_1)| = 0. \tag{6.13}$$

By the positivity of $\bar{\Phi}$, μ_* turns out to be the linearized first eigenvalue of V. Therefore we conclude that

$$\mu_* = \lambda_1(V). \tag{6.14}$$

In either of the Cases 1 or Case 2, we have the following by (6.7) and (6.14).

LEMMA 6.2.

$$\lim_{m\to\infty} \mu_1(v_{\sigma_m}, \Omega(\sigma_m)) \geqslant \min\{\mu_1(w_1, D_1), \mu_1(w_2, D_2), \lambda_1(V)\}.$$

We have completed the proof of Theorem 3 by Lemmas 6.1 and 6.2.

7. Structure of the Solutions

In Theorems 2 and 3, we have characterized the behavior of the solutions. In this section, we consider the behavior of their structures. We assume the following assumption besides the situation of Section 3.

(VIII.1) $\{\mu_k(w_i, D_i)\}_{k=1}^{\infty} \not\ni 0$ $(i=1,2)$, $\{\lambda_k(V)\}_{k=1}^{\infty} \not\ni 0$ for any w_1, w_2 and V which satisfy (3.2) and (3.3). By the similar method as in the proof of Theorem 3, we can prove that $\{\mu_k(v, \Omega(\zeta))\}_{k=1}^{\infty} \not\ni 0$ for any solution v of (3.1) (for small $\zeta > 0$). We consider the pair of the parameter ζ and the solution v of (3.1) in the space $X = \bigcup_{\zeta>0} \{\zeta\} \times C^{\infty}(\overline{\Omega(\zeta)})$. By the Implicit Function Theorem, the set of the solution pair (ζ, v) of (3.1) is locally a smooth curve in X when $\zeta > 0$ is small. Let K_ζ be all the solutions of (3.1) for ζ. By the above argument, K_{ζ_1} and K_{ζ_2} for small $\zeta_1, \zeta_2 > 0$ are mutually equivalent because any element v of K_ζ does not bifurcate by the implicit function theorem when the small parameter ζ varies. Therefore applying Theorem 2, we can imbed K_ζ into the following set for small $\zeta > 0$, $Y = \{(w_1, w_2, V) \in C^{\infty}(\bar{D}_1) \times C^{\infty}(\bar{D}_2) \times C^{\infty}([-1, 1]) \mid w_1, w_2$ and V satisfy the relations (3.2) and (3.3)$\}$.

Therefore we roughly regard Eq. (3.1) in the same manner as we do Eqs. (3.2) and (3.3) for small $\zeta > 0$.

8. Appendix

In this section, we mention an auxiliary proposition which is necessary in Sections 5 and 6 and review Matano's result in [9] which we used in Section 2. We assume $n \geqslant 2$ in Proposition 8.1.

Proposition 8.1. *Let φ be a function in $H^1(\Sigma)$ which satisfies the following conditions* (8.1) *and* (8.2), *where the set $\Sigma = \{(x_1, x') \in \mathbb{R}^n \mid x_1 > 0, |x| < c\}$ $(c > 0)$ and g is a real valued smooth function on $\mathbb{R}$:*

$$\Delta\varphi + g(\varphi) = 0 \qquad \text{in } \Sigma, \tag{8.1}$$

$$\frac{\partial\varphi}{\partial x_1} = 0 \qquad \text{on} \quad \partial\Sigma \cap \{x_1 = 0\} - \{0\}. \tag{8.2}$$

Then φ belongs to $C^{\infty}(\Sigma \cup (\{x_1 = 0\} \cap \partial\Sigma))$. In particular, the boundary condition is extended up to $\partial\Sigma \cap \{x_1 = 0\}$.

We give an outline of the proof of Proposition 8.1. We reflect φ about the hyperplane $x_1 = 0$ and obtain a solution φ_* of (8.1) (by using the boundary condition (8.2)) in the domain $\Sigma_* \equiv \{x \in \mathbb{R}^n \mid |x| < c\}$, i.e., we have

$$\varphi_* \in H^1(\Sigma_* - \{0\}), \tag{8.3}$$

$$\int_{\Sigma_*} (\nabla\varphi_* \nabla\gamma - g(\varphi_*)\gamma)\, dx = 0 \qquad \text{for any} \quad \gamma \in C_0^{\infty}(\Sigma_* - \{0\}). \tag{8.4}$$

On the other hand, from the following Lemma 8.1, we have that $C_0^\infty(\Sigma_* - \{0\})$ is dense in $H_0^1(\Sigma_*)$ and $H^1(\Sigma_* - \{0\}) = H^1(\Sigma_*)$ and Eq. (8.4) also holds for any $\gamma \in H^1(\Sigma_*)$. Therefore we have $\Delta\varphi_* + g(\varphi_*) = 0$ in Σ_* in weak sense. But, from the elliptic theory and the bootstrap argument, φ_* turns out to be a smooth function on Σ_*. Thus we obtain Proposition 8.1.

LEMMA 8.1 (Chavel–Feldman [2]). *Let X be a domain in $\mathbb{R}^n$ and Y be a smooth closed submanifold of X whose codimension in X is equal to or larger than 2 (Y may be a point which we regard as the 0-dimensional manifold). Then, for any $\gamma \in H^1(\Sigma_*)$, there exists a sequence $\{\gamma_m\}_{m=1}^\infty \subset C^\infty(X)$ such that*

$$\operatorname{supp} \gamma_m \subset X - Y \text{ for } m \geqslant 1 \qquad \text{and} \qquad \lim_{m \to \infty} \|\gamma_m - \gamma\|_{H^1(X)} = 0.$$

DEFINITION 2. A closed set $E \subset C^1(\bar{\Omega}) \cap C^2(\Omega)$ is said to be positively invariant under (1.7), if, given any $w \in E$, the solution $u(t, x)$ of (1.7) with initial data $u(0, \cdot) = w(\cdot)$ is defined globally on $[0, \infty) \times \Omega$ and satisfies $u(t, \cdot) \in E$ for all $t \geqslant 0$.

PROPOSITION 8.2 (Matano [9]). *Let $E, E_1, E_2, E_3, \ldots$ be a family of nonempty sets in $C^1(\bar{\Omega}) \cap C^2(\Omega)$ such that*

(a) *$E_1 \supset E_2 \supset E_3 \supset \cdots$ and $\bigcap_{m=1}^\infty E_m = E$;*

(b) *each E_m is closed in $C^1(\bar{\Omega}) \cap C^2(\Omega)$ and bounded in $L^\infty(\Omega)$; moreover, for each m, E_{m+1} is contained in the interior of E_m with respect to the topology of $C^1(\bar{\Omega}) \cap C^2(\Omega)$;*

(c) *each E_m is positively invariant under* (1.7).

Then E contains at least one stable solution of (1.1).

Remark. In [9], Matano established Proposition 8.2 and first constructed a nonconstant stable solution of (1.1) for the domain similar to $\Omega(\zeta)$ in Section 2 and later generalized it to the case of the dynamical system which has the strong order preserving property and also constructed a nonconstant stable equilibrium solution of the competition diffusion system (cf. Matano and Mimura [11]). Theorem 1 in this paper can be regarded as a much improved version of Matano's existence result of the nonconstant stable solution for the single reaction diffusion equation.

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